

Bicevian Conics and CPCC Cubics

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1 Definitions and general theorems

1.1 Notations

Let $P = u : v : w$ and $Q = p : q : r$ be two points in the plane of reference triangle ABC and let $P_aP_bP_c$, $Q_aQ_bQ_c$ be their respective cevian triangles. We denote by :

- $P \times Q$ the barycentric product of P and Q .¹
- $\mathbb{P}(P)$, $\mathbb{P}(Q)$ the trilinear polars of P , Q ,
- $\mathbb{C}(P)$, $\mathbb{C}(Q)$ the circum-conics with perspectors P , Q ,
- $\mathbb{I}(P)$, $\mathbb{I}(Q)$ the in-conics with perspectors P , Q ,
- $\mathcal{D}(P, Q)$ the circum-conic through P and Q ,
- $\mathcal{E}(P, Q)$ the diagonal conic through P and Q which contains the vertices of the anticevian triangles of P , Q ,
- $Y = P \ominus Q$ the perspector of $\mathcal{D}(P, Q)$. This is the intersection of $\mathbb{P}(P)$ and $\mathbb{P}(Q)$ with coordinates

$$pu(rv - qw) :: \quad \text{or} \quad \frac{1}{rv} - \frac{1}{qw} ::,$$

- $T = P \oplus Q$ the crosspoint of P and Q i.e. the pole of the line PQ in $\mathcal{D}(P, Q)$ or equivalently the intersection of the tangents at P , Q to this conic. This has coordinates

$$pu(rv + qw) :: \quad \text{or} \quad \frac{1}{rv} + \frac{1}{qw} ::,$$

¹This point can easily be related to the cevian triangles of P and Q . Indeed, let us call Ω_a the central point of the involution on BC which swaps B and C , P_a and Q_a (in other words, the foot on BC of the radical axis of the circles with diameters BC and P_aQ_a). Ω_b , Ω_c are defined similarly and the three points lie on a same line whose trilinear pole is $P \times Q$.

- P/Q the cevian quotient of P and Q (or P -Ceva conjugate of Q) i.e. the perspector of the cevian triangle of P and the anticevian triangle of Q . It is also the pole of $\mathbb{P}(P)$ in $\mathbb{C}(Q)$ with coordinates

$$P/Q = p \left(-\frac{p}{u} + \frac{q}{v} + \frac{r}{w} \right) ::,$$

- $R = P \otimes Q$ the cevian product (or cevapoint) of P and Q namely the point

$$R = \frac{1}{rv + qw} ::,$$

such that

$$R = P \otimes Q \iff P = R/Q \iff Q = R/P.$$

$P \otimes Q$ is the pole of the line PQ in $\mathcal{E}(P, Q)$. This conic has equation

$$\sum_{\text{cyclic}} (rv - qw)(rv + qw) x^2 = 0$$

and center

$$R \times Z = \frac{1}{(rv - qw)(rv + qw)} ::,$$

where Z is the trilinear pole of the line PQ i.e. the fourth intersection of the conics $\mathbb{C}(P)$, $\mathbb{C}(Q)$, namely the point

$$Z = \frac{1}{rv - qw} ::.$$

1.2 Theorems

Theorem 1. *There is a unique conic passing through the six points $P_a, P_b, P_c, Q_a, Q_b, Q_c$ which we call the bicevian conic of P and Q and denote by $\mathcal{C}(P, Q)$.*

Proof. Let $\mathcal{F}$ be the pencil of conics passing through P_b, P_c, Q_b, Q_c . Each conic of $\mathcal{F}$ meets BC at M, N which are homologous in a same involution. In particular B, C and the feet P'_a, Q'_a of $\mathbb{P}(P), \mathbb{P}(Q)$ on BC are homologous. Hence, the harmonic conjugates of P'_a, Q'_a with respect to B, C are also homologous but these points are P_a, Q_a . This insures there is a conic of $\mathcal{F}$ passing through these two latter points. $\square$

Remarks :

- another proof can be derived from Carnot's and Ceva's theorems.
- the pole of the line PQ in $\mathcal{C}(P, Q)$ is Y .
- recall there is also a unique conic passing through the vertices of the anticevian triangles of P and Q . This is $\mathcal{E}(P, Q)$.
- by duality, there is a unique conic inscribed in both cevian triangles of P and Q . This is the conic $\mathbb{I}(P, Q)$ with equation

$$\sum_{\text{cyclic}} \frac{x^2}{p^2 u^2 (rv - qw)(rv + qw)} = 0$$

with center $T \times Y$.

Note that the triangle ABC is perspective to the triangles formed by the contacts of $\mathbb{I}(P, Q)$ with the sidelines of the anticevian triangles of P and Q at the respective points

$$p^2 u (rv - qw)(rv + qw) :: \quad \text{and} \quad pu^2 (rv - qw)(rv + qw) :: ,$$

both lying on $\mathbb{P}(T)$.

Theorem 2. $\mathcal{C}(P, Q)$ is the nine point conic of the quadrilateral $ABCP$ in an affine map of the projective plane where the line at infinity is $\mathbb{P}(Q)$ or vice-versa. Equivalently, $\mathcal{C}(P, Q)$ is the locus of poles of $\mathbb{P}(Q)$ (resp. $\mathbb{P}(P)$) in the circum-conics passing through P (resp. Q)².

Corollary 1. $\mathcal{C}(P, Q)$ is the locus of X for which the cevian quotient P/X lies on $\mathbb{P}(Q)$ and vice-versa.

In other words, for any point X on $\mathbb{P}(Q)$ (resp. $\mathbb{P}(P)$), P/X (resp. Q/X) is a point on $\mathcal{C}(P, Q)$.

Proof. P/X can be seen as the Q -center of $\mathcal{C}(P, Q)$. Recall that P/X is also the pole of $\mathbb{P}(P)$ in $\mathbb{C}(X)$.

When $Q = G$, $\mathbb{P}(Q) = \mathcal{L}^\infty$ and this is the “true” center. □

Corollary 2. $\mathcal{C}(P, Q)$ also passes through P if and only if Q lies on the circum-conic with perspector P .

Proof. P lies on $\mathcal{C}(P, Q)$ if and only if $P/P = P$ lies on $\mathbb{P}(Q)$. This is equivalent to Q lies on $\mathbb{C}(P)$. □

²This is in fact a very special case of a more general result : $\mathcal{C}(P, Q)$ is the mixed polococonic of $\mathbb{P}(P)$ and $\mathbb{P}(Q)$ in the degenerate cubic formed by the sidelines of ABC . See [1], §2.3.5.

1.3 Consequences

Following Corollary 1, we can find infinitely many points on $\mathcal{C}(P, Q)$. Some of them are closely related to the geometry of the configuration i.e. to the intersections of $\mathbb{P}(P)$ and $\mathbb{P}(Q)$ with several particular lines.

Apart the six points $P_a, P_b, P_c, Q_a, Q_b, Q_c$, the bicevian conic $\mathcal{C}(P, Q)$ contains :

1. the six harmonic conjugates of the intersections of $\mathbb{P}(P)$ (resp. $\mathbb{P}(Q)$) and the sidelines of the quadrilateral $ABCQ$ (resp. $ABCP$) with respect to the two vertices on the sideline.

For example, AP meets $\mathbb{P}(Q)$ at a point whose harmonic conjugate with respect to A, P is a point on $\mathcal{C}(P, Q)$.

2. the pole P_1 of $\mathbb{P}(P)$ in the conic $\mathcal{D}(P, Q)$ i.e. $pu^2(rv - qw)^2 ::$, the fourth intersection of $\mathcal{C}(P, Q)$ and $\mathbb{I}(P)$ and also $Q/(P \ominus Q)$,
similarly, the pole P_2 of $\mathbb{P}(Q)$ in the conic $\mathcal{D}(P, Q)$ i.e. $p^2u(rv - qw)^2 ::$, the fourth intersection of $\mathcal{C}(P, Q)$ and $\mathbb{I}(Q)$ and also $P/(P \ominus Q)$,

remark : the line P_1P_2 is the last common tangent of the in-conics $\mathbb{I}(P)$ and $\mathbb{I}(Q)$. It is the trilinear polar of $P \ominus Q$. Note also that this point is the pole of the line PQ in $\mathcal{C}(P, Q)$.

3. the pole P_3 of $\mathbb{P}(P)$ in the circum-conic passing through Q , tangent at Q to PQ i.e. $p^2(rv - qw)^2(pruv + pquw + p^2vw - qru^2) ::$,
similarly, the pole P_4 of $\mathbb{P}(Q)$ in the circum-conic passing through P , tangent at P to PQ i.e. $u^2(rv - qw)^2(pruv + pquw - p^2vw + qru^2) ::$.
remark : recall that the two circum-conics $\mathbb{C}(P), \mathbb{C}(Q)$ have a fourth common point $Z = \frac{1}{rv - qw} ::$ which is the trilinear pole of the line PQ . Z/Q is the intersection of $\mathbb{P}(Q)$ and $\mathbb{P}(R)$ and $P_3 = P/(Z/Q)$. Similarly, $P_4 = Q/(Z/P)$.

4. points related to the line PQ .
5. points related to the tangents at P, Q to $\mathcal{D}(P, Q)$.
6. points related to the infinite points of $\mathbb{P}(P)$ and $\mathbb{P}(Q)$.

The equation of $\mathcal{C}(P, Q)$ is :

$$\sum_{\text{cyclic}} qrvwx^2 - pu(rv + qw)yz = 0 \quad (1)$$

or

$$\sum_{\text{cyclic}} \frac{x^2}{pu} - \left(\frac{1}{qw} + \frac{1}{rv} \right) yz = 0 \quad (2)$$

or

$$2 \sum_{\text{cyclic}} pu(rv + qw)yz - \left(\sum_{\text{cyclic}} qrx \right) \left(\sum_{\text{cyclic}} vwx \right) = 0 \quad (3)$$

or

$$\sum_{\text{cyclic}} pu(q + r)(v + w)yz - (x + y + z) \sum_{\text{cyclic}} qrvwx = 0. \quad (4)$$

The equations (1), (2) show that $\mathcal{C}(P, Q)$ belongs to the pencil of conics generated by the diagonal conic Γ_d ³ and the circum-conic $\Gamma_c = \mathbb{C}(T)$ with respective equations

$$\sum_{\text{cyclic}} qrvwx^2 = 0 \quad \text{and} \quad \sum_{\text{cyclic}} pu(rv + qw)yz = 0 \quad (5)$$

which meet at four (real or not) points which are the fixed points of the involution on $\mathbb{P}(Q)$ (resp. $\mathbb{P}(P)$) generated by the pencil of circum-conics passing through P (resp. Q).

The equation (3) shows that these four points lie on Γ_c (and naturally on $\mathbb{P}(P)$ and $\mathbb{P}(Q)$).

The equation (4) shows that $\mathcal{C}(P, Q)$ meets $\mathcal{L}^\infty$ and $\mathbb{P}(P \times Q)$ ⁴ at the same points as the circum-conic $\Gamma(P, Q)$ with equation

$$\sum_{\text{cyclic}} pu(q + r)(v + w)yz = 0 \quad (6)$$

whose perspector $S = pu(q + r)(v + w) ::$ is the barycentric product of the centers of the two in-conics $\mathbb{I}(P)$ and $\mathbb{I}(Q)$, these centers being the complements of the isotomic conjugates of P and Q . In other words, the conics $\mathcal{C}(P, Q)$ and $\Gamma(P, Q)$ are homothetic. See figure 1.

Remarks :

1. when Q is the centroid of ABC , $\mathbb{P}(Q)$ is the line at infinity $\mathcal{L}^\infty$ hence the conic $\mathcal{C}(P, G)$ contains the midpoints of BC , CA , AB , PA , PB , PC .

³The center of Γ_d is $P \times Q$.

⁴Note that these three trilinear polars $\mathbb{P}(P \times Q)$, $\mathbb{P}(P)$, $\mathbb{P}(Q)$ concur if and only if G , P , Q are collinear.



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with respect to the medial triangle $M_a M_b M_c$ of $P_a P_b P_c$ of the conic inscribed in $P_a P_b P_c$ with perspector P .

$\mathcal{Q}(P)$ has three nodes M_a, M_b, M_c and passes through the traces U, V, W of $\mathbb{P}(P)$. The fourth point on $\mathbb{P}(P)$ is $u(v-w)(2u+v+w) ::$ obtained when Q is the infinite point of $\mathbb{P}(P)$.

The tangents at U, V, W are the sidelines of $P_a P_b P_c$. See figure 2.

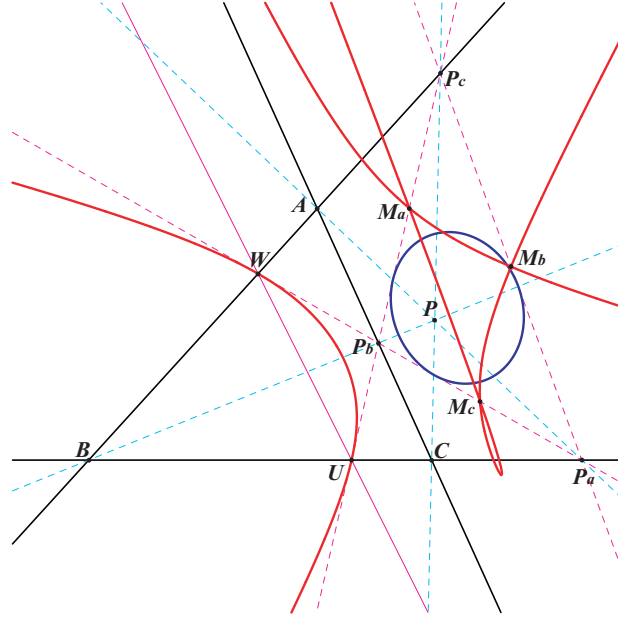


Figure 2: The quartic $\mathcal{Q}(P)$

1.4 The Euler case

We illustrate these general considerations with the nine point circle i.e. the bicevian conic $\mathcal{C}(G, H)$. Here, $\mathbb{P}(G)$ is the line at infinity and $\mathbb{P}(H)$ is the orthic axis. The related in-conics are the Steiner in-ellipse $\mathbb{I}(G)$ and the K -ellipse $\mathbb{I}(H)$.

$\mathcal{C}(G, H)$ is the locus of centers of rectangular circum-hyperbolas and also the locus of poles of the orthic axis in the circum-conics passing through G .

The aforementioned points are :

$P_1 = X_{115}$, the center of the Kiepert hyperbola, a point on $\mathbb{I}(G)$,

$P_2 = X_{125}$, the center of the Jerabek hyperbola, a point on $\mathbb{I}(H)$,

$$P_3 = X_{136},$$

$$P_4 = X_{127}.$$

The harmonic conjugates we met in consequence 1 are the midpoints of AH , BH , CH and the second intersections with the medians of ABC .

The conics Γ_d and Γ_c are the polar circle and the circum-circle. See figure 3.

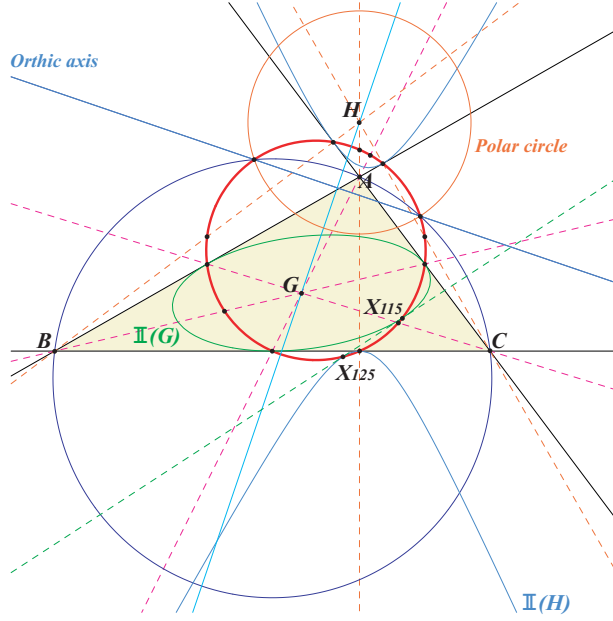


Figure 3: The Euler case

1.5 Bitangent conics

Following Theorem 2 above, the polar line in $\mathcal{C}(P, Q)$ of $Y = P \ominus Q$ (the intersection of $\mathbb{P}(P)$, $\mathbb{P}(Q)$ and the perspector of the circum-conic through P and Q) is the line PQ which meets $\mathcal{C}(P, Q)$ at two points Z_1, Z_2 such that the tangents to $\mathcal{C}(P, Q)$ at these points pass through Y .

Now, if M is any point on PQ and if M_P (resp. M_Q) is the pole of $\mathbb{P}(P)$ (resp. $\mathbb{P}(Q)$) in the circum-conic through M and Q (resp. P), the line $M_P M_Q$ envelopes a conic inscribed in ABC and bitangent at Z_1, Z_2 to $\mathcal{C}(P, Q)$. Its perspector is $R = P \otimes Q$, the cevian product (or cevapoint) of P and Q .

In other words, for any bicevian conic $\mathcal{C}(P, Q)$, there is an in-conic bitangent to $\mathcal{C}(P, Q)$ at the points Z_1, Z_2 where $\mathcal{C}(P, Q)$ meets the line PQ . See figure 4. The center of the in-conic is the point $u(q + r) + p(v + w) ::$ (complement of the isotomic conjugate of R) which is in general different of that of $\mathcal{C}(P, Q)$. See §2.5.

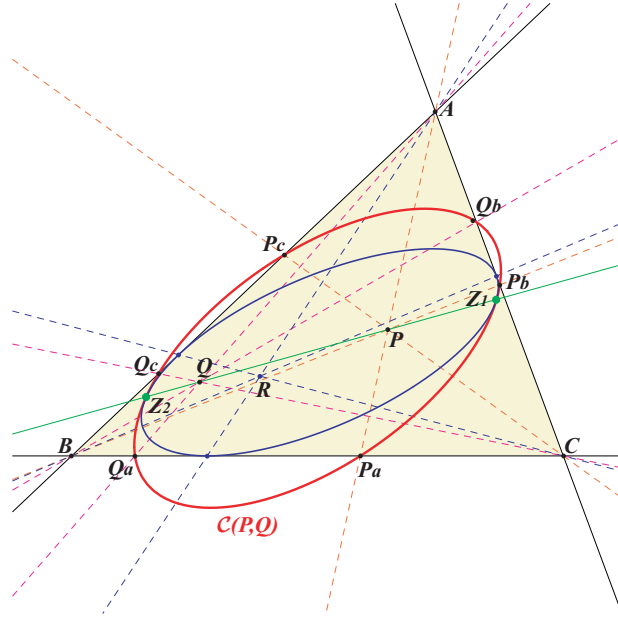


Figure 4: Bitangent conics

For example, the nine point circle $\mathcal{C}(G, H)$ is bitangent to the McBeath conic at two points on the Euler line which are X_{1312}, X_{1313} and the de Longchamps ellipse $\mathcal{C}(I, X_{57})$ is bitangent to the incircle at X_{2446}, X_{2447} on the line IO (in both cases, the two conics have the same center).

Conversely, any in-conic $\mathbb{I}(R)$ with perspector R is bitangent to all the bicevian conics $\mathcal{C}(P, Q)$ such that R is the cevapoint of P and Q i.e. such that Q is the cevian quotient (Ceva conjugate) of R and P . The contacts of the two conics lie on the line PQ . This line passes through the center of $\mathbb{I}(R)$ if and only if R lies on the pivotal cubic with pivot R which is invariant in the isoconjugation swapping the perspector R and the center of $\mathbb{I}(R)$ i.e. the isopivot R^* of the cubic. See CL042 in [2]. Hence, the cubic is the locus of M such that R^*, M and R/M are collinear. From this, any line through R meets the cubic at two other (real or not) points P, Q such that

$\mathcal{C}(P, Q)$ is the bicevian conic bitangent to $\mathbb{I}(R)$ at the two points where the line meets $\mathbb{I}(R)$. Note that, when the two points are not real, the bicevian conic remains real but not the intersections with the sidelines of ABC .

For example, the incircle is bitangent to $\mathcal{C}(X_i, X_j)$ for $(i, j) \in \{(1, 57), (2, 145), (4, 196), (11, 514), (55, 218), (56, 222), (63, 224), (65, 226), (81, 229), (174, 2089), (177, 234)\}$. The pivotal cubic is **K365** = $p\mathcal{K}(X_{57}, X_7)$ passing through $X_1, X_2, X_7, X_{57}, X_{145}, X_{174}, X_{1488}, X_{2089}$ hence the bicevian conics $\mathcal{C}(X_i, X_j)$ for $(i, j) \in \{(1, 57), (2, 145), (174, 2089)\}$ are bitangent to the incircle at two points which are antipodes on this circle. See figure 5 in which the two situations are represented : real intersections (in blue) and not real intersections (in green).

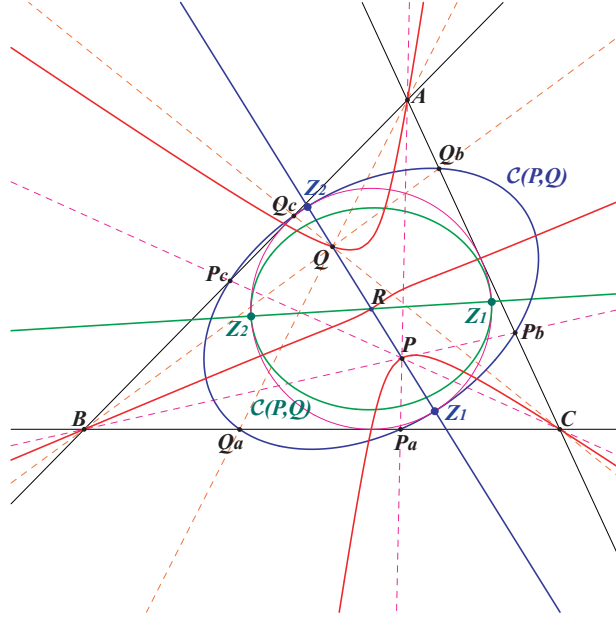


Figure 5: Bitangent conics (reciprocal)

1.6 Antipodes on $\mathcal{C}(P, Q)$

We now generalize the special cases we have met above.

We know that, for any points M_Q and M'_Q on $\mathbb{P}(Q)$, the points $X = P/M_Q$ and $X' = P/M'_Q$ are two points on $\mathcal{C}(P, Q)$.

These points X and X' are antipodes on $\mathcal{C}(P, Q)$ if and only if M_Q and M'_Q are conjugated in $\mathcal{C}(G, P)$.

Consequently, the points M_P and M'_P on $\mathbb{P}(P)$ such that $M_P = Q/X$ and $M'_P = Q/X'$ are conjugated in $\mathcal{C}(G, Q)$.

Remarks :

- these two conics $\mathcal{C}(G, P)$ and $\mathcal{C}(G, Q)$ pass through the vertices of the medial triangle and meet at a fourth point which is G/Y .
- if P and Q are distinct of G , $\mathcal{C}(P, Q)$ and $\mathcal{C}(G, P)$ meet at the vertices of the cevian triangle of P and a fourth point which is the P -Ceva conjugate of the infinite point of $\mathbb{P}(Q)$.
- the lines $M_P M'_Q$ and $M_Q M'_P$ meet at a point on $\mathbb{P}(T)$.
- the lines $M_P M_Q$ and $M'_P M'_Q$ also meet at a point lying on a fixed line and envelope the conic $\mathbb{I}(P, Q)$.⁵ Note that these lines $M_P M_Q$ and $M'_P M'_Q$ are the polar lines of X and X' in $\mathcal{D}(P, Q)$ and the conics $\mathcal{C}(P, Q)$ and $\mathbb{I}(P, Q)$ are dual with respect to $\mathcal{D}(P, Q)$.

2 Perspector, center and foci of bicevian conics

2.1 Perspector

We borrow this idea to Floor van Lamoen.

Let A_1 be the intersection of the tangents at P_a, Q_a to $\mathcal{C}(P, Q)$ and define B_1, C_1 similarly. The triangles ABC and $A_1 B_1 C_1$ are perspective at the point $\mathcal{P}(P, Q)$ called the perspector of $\mathcal{C}(P, Q)$. The coordinates are :

$$\frac{pu}{p^2vw + qru^2 + 3pu(rv + qw)} :: .$$

$\mathcal{P}(P, Q)$ is also the perspector of ABC and the polar triangle of ABC in $\mathcal{C}(P, Q)$.⁶

Obviously, when $P = Q$ this point is P , the perspector of the in-conic $\mathcal{C}(P, P) = \mathbb{I}(P)$ and $\mathbb{P}(P)$ is its perspectrix.

For example, the perspector of the nine point circle is the point with coordinates $\frac{1}{2a^2 S_A + b^2 c^2} ::$, the isotomic conjugate of X_{1078} .

⁵recall that $\mathbb{I}(P, Q)$ is inscribed in both cevian triangles of P and Q , that it is tangent to $\mathbb{P}(P)$ and $\mathbb{P}(Q)$ at two points on $\mathbb{P}(T)$ and that its center is the barycentric product of Y and T .

⁶i.e. the triangle formed by the three polar lines of A, B, C in $\mathcal{C}(P, Q)$. Note that the polar line of A passes through the intersection A_2 of the lines $P_b P_c$ and $Q_b Q_c$. If B_2, C_2 are defined similarly, the triangles $ABC, A_1 B_1 C_1$ and $A_2 B_2 C_2$ are triply perspective.

2.2 $\mathcal{C}(P, Q)$ with center P

The center of the bicevian conic $\mathcal{C}(P, Q)$ is rather complicated and we shall only investigate a couple of particular cases.

Theorem 3. *There is a unique bicevian conic with center P which we denote by $\mathcal{CC}(P)$.*

Proof. $\mathcal{C}(P, Q)$ has center P if and only if $Q = u/(v + w - u) ::$ in which case $\mathbb{P}(P)$ and $\mathbb{P}(Q)$ are parallel.⁷ This point Q is the barycentric product of P and the perspector of the inconic with center P . $\square$

The barycentric equation of $\mathcal{CC}(P)$ is :

$$\sum_{\text{cyclic}} [(v + w - u) v^2 w^2 x^2 - 2 u^3 v w y z] = 0$$

or equivalently :

$$\sum_{\text{cyclic}} \left[(v + w - u) \left(\frac{x}{u} \right)^2 - 2 u \frac{y}{v} \frac{z}{w} \right] = 0$$

Remarks

– $\mathcal{CC}(P)$ is an ellipse, a parabola, a hyperbola if and only if the quantity $uvw(u + v + w)$ is positive, null, negative respectively.

– $\mathcal{CC}(P)$ is bitangent to the in-conic with center P and the two conics are said to be centrally associated. The line passing through the two points of tangency T_1, T_2 is the conjugated diameter (with respect to the conics) of the trilinear polar of P . See figure 6. Note that this line also contains P and Q and has equation

$$\sum_{\text{cyclic}} (v + w - u)(v - w) v w x = 0.$$

– Let us denote by

$$\phi_P : X(x : y : z) \mapsto X' \left(-\frac{x}{u} + \frac{y}{v} + \frac{z}{w} : \frac{x}{u} - \frac{y}{v} + \frac{z}{w} : \frac{x}{u} + \frac{y}{v} - \frac{z}{w} \right)$$

the mapping associating to the point X the anticomplement of the barycentric product of X and the isotomic conjugate of P .

⁷the locus of Q for which $\mathbb{P}(P)$ and $\mathbb{P}(Q)$ are parallel is the circum-conic with perspector the infinite point of $\mathbb{P}(P)$. Thus, this conic passes through G and is centered on the inscribed Steiner ellipse.

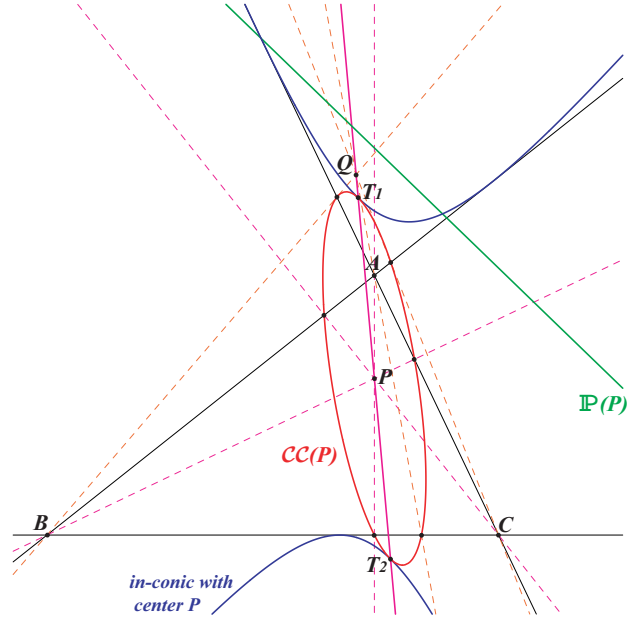


Figure 6: $\mathcal{CC}(P)$ or $\mathcal{C}(P, Q)$ with center P

ϕ_P transforms $\mathcal{CC}(P)$ into $\mathbb{C}(P)$ and the in-conic with center P into another conic bitangent to $\mathbb{C}(P)$, the points of tangency lying on the line PQ .

For example, when we take $P = I$, the conic $\mathcal{CC}(I)$ is an ellipse centrally associated to the incircle called the de Longchamps ellipse. Thus, ϕ_I transforms this ellipse into $\mathbb{C}(I)$, the circum-conic with perspector I , center X_9 (mittenpunkt). The point Q is X_{57} . See figure 7.

Note that this de Longchamps ellipse is also the supplement of the circum-circle.

The de Longchamps ellipse is bitangent to the incircle at X_{2446} and X_{2447} (these are the endpoints of the minor axis). The major axis is S_1S_2 and $S_1S_2 = \sqrt{\frac{abc}{a+b+c}}$.

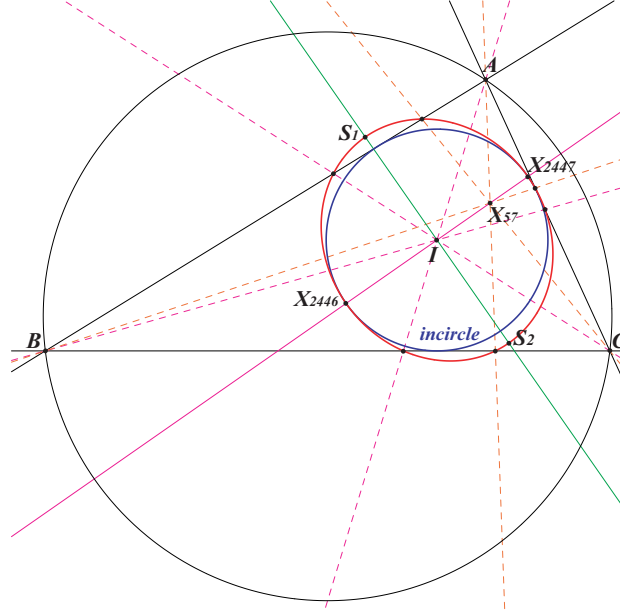


Figure 7: The de Longchamps ellipse

2.3 $\mathcal{C}(P, Q)$ with center the midpoint of PQ

2.3.1 The general case

Theorem 4. *For a given P , there are exactly six points Q such that $\mathcal{C}(P, Q)$ has center the midpoint of PQ .*

Note that three of these points give degenerate conics and the three other might not be all real.

Indeed, when we write that the center of $\mathcal{C}(P, Q)$ lies on the line PQ , we obtain a condition which can be construed as either :

- (1) Q lies on the circum-conic Γ_P passing through G and P .⁸
- (2) Q lies on the sidelines of the anticevian triangle of P .

In case (1), for a given point P and for a variable point Q on Γ_P , the locus of the center of the bicevian conic $\mathcal{C}(P, Q)$ is another conic which contains P , the center of $\mathbb{I}(P)$, the vertices of the anticevian triangle of the

⁸This conic is the locus of point X such that P , X and the barycentric product of P , X are collinear. It is the circum-conic with perspector the infinite point of $\mathbb{P}(P)$.

perspector of Γ_P ⁹. These bicevian conics form a pencil of conics passing through P_a, P_b, P_c and the pole of $\mathbb{P}(P)$ in Γ_P ¹⁰.

It follows that the sought points Q must lie on the homothetic of this conic under $h(P, 2)$ and on Γ_P which gives at most three points Q , one at least being real. These two conics generate a pencil which also contains the conic with equation

$$\sum_{cyclic} u(v-w)(vwx^2 - u^2yz) = 0.$$

The construction is generally not possible with ruler and compass only since it needs to intersect two conics, one of them being Γ'_P , the homothetic of Γ_P under $h(P, 1/2)$. The other conic Γ''_P contains P , the center of the in-conic with perspector P ¹¹, the vertices of the tangential triangle of Γ_P ¹². These conics have P and three other points in common which are the centers of the sought bicevian conics. The reflections of P in these points are the sought points Q . The three corresponding bicevian conics belong to the same pencil of conics since they have four common points, namely P_a, P_b, P_c and F , the center of the circum-conic passing through P and its isotomic conjugate. Note that this center F lies on $\mathbb{I}(P)$.

The figure 8 shows a case ($P = H$) where one can find three real $\mathcal{C}(P, Q)$.

Remark : these three conics have the same directions of axes if and only if the pencil of conics through P_a, P_b, P_c and F contains a circle i.e. if and only if P lies either on the Lucas cubic or on the Tucker nodal cubic K015. This is the case in figure 8 since $P = H$.

In case (2), there are three (all real) points Q but the corresponding bicevian conics $\mathcal{C}(P, Q)$ all degenerate into two lines. These points are the common points of the sidelines of the anticevian triangle of P and the trilinear polar of the barycentric quotient of P and its complement, namely the point $u/(v+w)^2 ::$.

2.3.2 A special case

Paul Yiu (private message, January 27, 2005) has found a special situation where the construction is possible with ruler and compass.

⁹these three points are the midpoints of a vertex of ABC and the foot of the parallel at this vertex to $\mathbb{P}(P)$.

¹⁰this is the point $u(v-w)^2 ::$.

¹¹This is the complement of the isotomic conjugate of P .

¹²i.e. the triangle formed by the tangents at A, B, C to Γ_P .

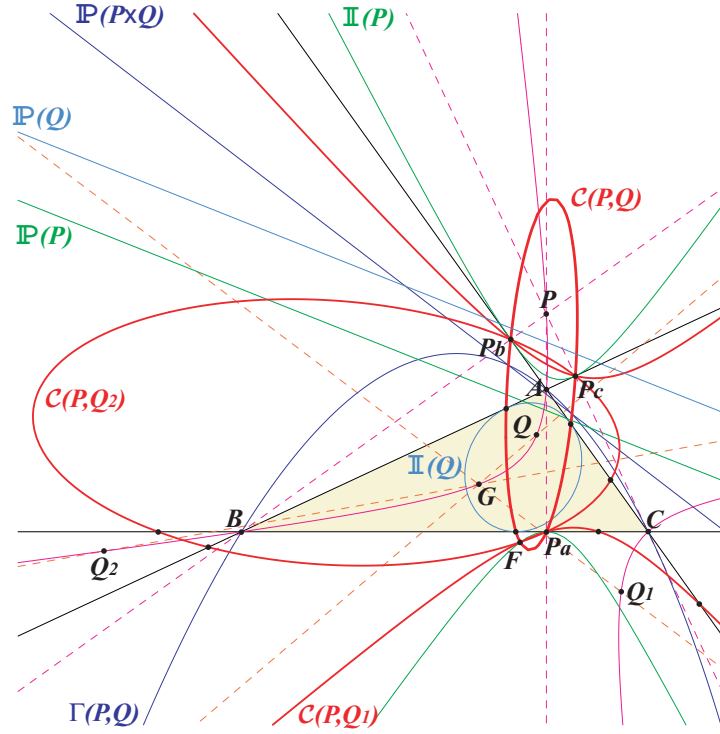


Figure 8: $\mathcal{C}(P, Q)$ with center the midpoint of P, Q

Theorem 5 (Yiu). *For an arbitrary infinite point $f : g : h$, we take P and Q to be the tripolar centroids¹³ of the points $1/f : 1/g : 1/h$ and $1/(g - h) : 1/(h - f) : 1/(f - g)$. Then the bicevian conic $\mathcal{C}(P, Q)$ has center the midpoint of PQ , which is also the barycentric product of P and Q .*

We add the following remarks (see figure 9) :

- these two points P, Q are naturally on the same circum-conic through G ,
- the line PQ is now tangent to the Steiner in-ellipse,
- P, Q also lie on the homothetic of the Tucker nodal cubic K015 under $h(G, 1/2)$ or the reflection of K219 in G ,
- $P \times Q$ (the center of $\mathcal{C}(P, Q)$) lies on K219, the complement of K015.

¹³The tripolar centroid of a point $P = u : v : w \neq G$ is the isobarycenter of the three (collinear) points of intersection of $\mathbb{P}(P)$ with the sidelines of ABC . It has barycentrics $u(v - w)(v + w - 2u) : :$

For example, with X_{523} , X_{514} , X_{525} at infinity, we obtain X_{1641} , X_{1644} , X_{1651} on cK015.

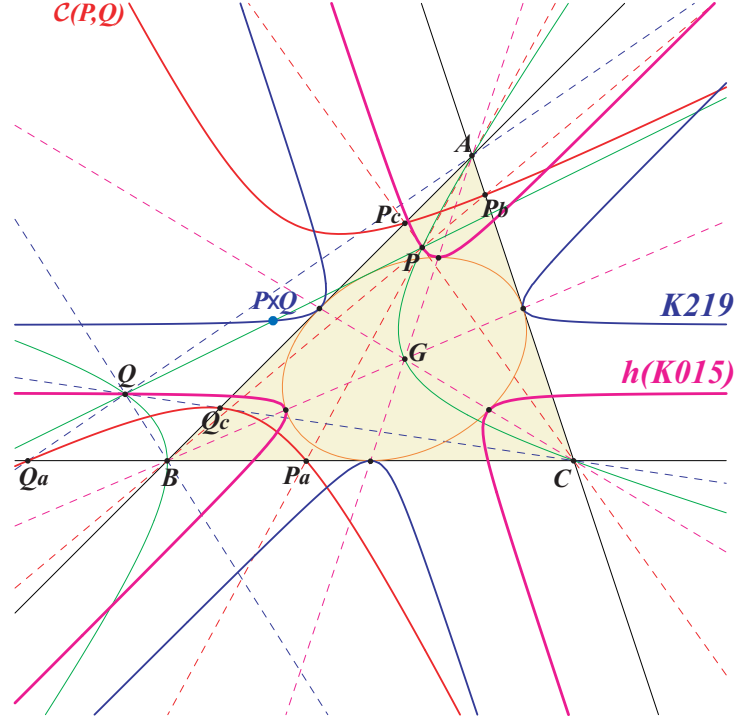


Figure 9: Special $\mathcal{C}(P, Q)$ with center $P \times Q$

2.4 $\mathcal{C}(P, Q)$ with focus P

Theorem 6. *For a given point P , there are in general four points Q such that $\mathcal{C}(P, Q)$ has focus P .*

Proof. If $\mathcal{C}(P, Q)$ has focus P , it is tangent to the isotropic lines through P and passes through the three given points P_a, P_b, P_c . It is a known fact that there are four conics given three points and two tangents. $\square$

Construction : first draw the circles passing through P with respective centers P_a, P_b, P_c . These circles have two by two six homothety centers which are three by three collinear on four lines. These lines are the directrices of the four sought conics. Thus, each conic is perfectly characterized since

we know the focus P , the relative directrix Δ and the three points P_a, P_b, P_c . Since the focal axis is the perpendicular at P to Δ , it is sufficient to reflect P_a, P_b, P_c in this axis to obtain three more points on the conic and then complete its construction.

It is easy to draw the tangents at P_a, P_b, P_c : the perpendicular at P to PP_a meets Δ at I_a which is a point on the tangent at P_a .

Figure 10 shows one of these four bicevian conics when P is the Nagel point X_8 .

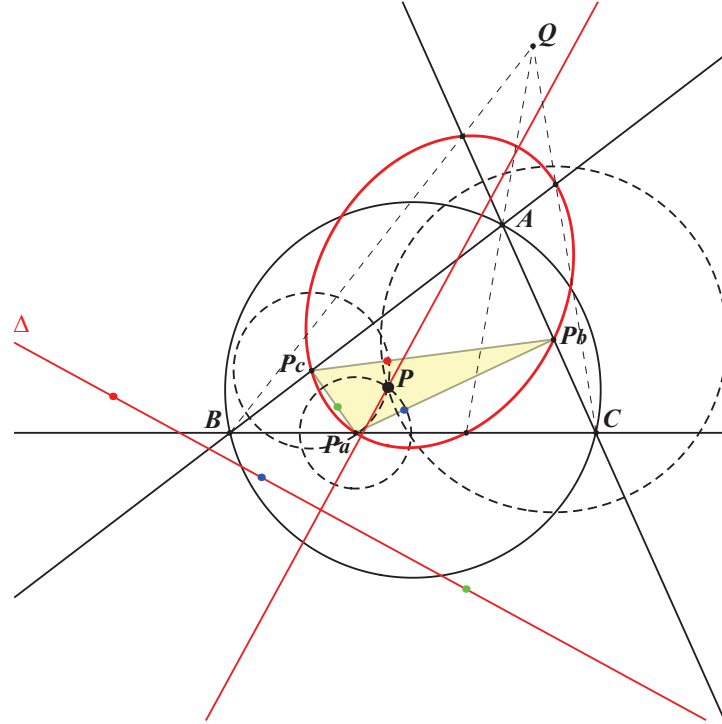


Figure 10: $\mathcal{C}(P, Q)$ with focus P

2.5 $\mathcal{C}(P, Q)$ and $\mathbb{I}(R)$ with same center

Recall that there is a unique inscribed conic $\mathbb{I}(R)$ which is bitangent to $\mathcal{C}(P, Q)$ at the two intersections of the line PQ with $\mathcal{C}(P, Q)$. Its perspector is $R = P \otimes Q$, the cevian product (or cevapoint) of P and Q and its center is the complement of the isotomic conjugate of R .

Theorem 7. $\mathcal{C}(P, Q)$ and $\mathbb{I}(R)$ share the same center if and only if either
(1) P and Q lie on a same circum-conic passing through the centroid G of ABC , or
(2) Q is one of the common points of three decomposed circum-cubics described below.

In case (1) we have the following properties.

- When one of the points P, Q is the centroid G of ABC , $\mathcal{C}(P, Q)$ and $\mathbb{I}(R)$ always share the same center.
- If P is a given point distinct of G and if Q is a variable point on the circum-conic passing through G and P then all the bicevian conics $\mathcal{C}(P, Q)$ are in a same pencil. They all contain the vertices of the cevian triangle of P and the center of the circum-conic passing through P and the isotomic conjugate P' of P . Naturally, this center also lies on $\mathcal{C}(G, P)$, $\mathbb{I}(P)$ and also $\mathbb{I}(P')$.
- This pencil contains a circle if and only if P lies on the Lucas cubic [K007](#) in which case the cevian triangle of P is a pedal triangle. Note that the circum-conic passing through G and P meets [K007](#) at a sixth point which is the cyclocevian conjugate of P . This latter point lies on the line through X_{20} and P .
- If U is a point distinct of G , let G/U be the center of the circum-conic with perspector U (i.e. the G -Ceva conjugate of U). For any two points P and Q harmonically conjugated with respect to U and G/U , the conics $\mathcal{C}(P, Q)$ and $\mathbb{I}(R)$ have the same center which lies on the line $U, G/U$. These points P, Q lie on a same circum-conic through G . The point R lies on the circum-conic through U and G/U .

The most interesting case is probably obtained with $U = X_6$ hence $G/U = X_3$. The points P, Q lie on the Brocard axis and are inversive images with respect to the Brocard circle and one can find quite many such pairs of points P, Q . Note that R is a point on the Jerabek hyperbola.

Figure 11 shows two of these bicevian conics and their corresponding inscribed conics namely :

- $\mathcal{C}(X_{15}, X_{16})$ and $\mathbb{I}(X_4)$ with same center X_6 ,
- $\mathcal{C}(X_{371}, X_{372})$ and $\mathbb{I}(X_{74})$ with same center X_{3003} .

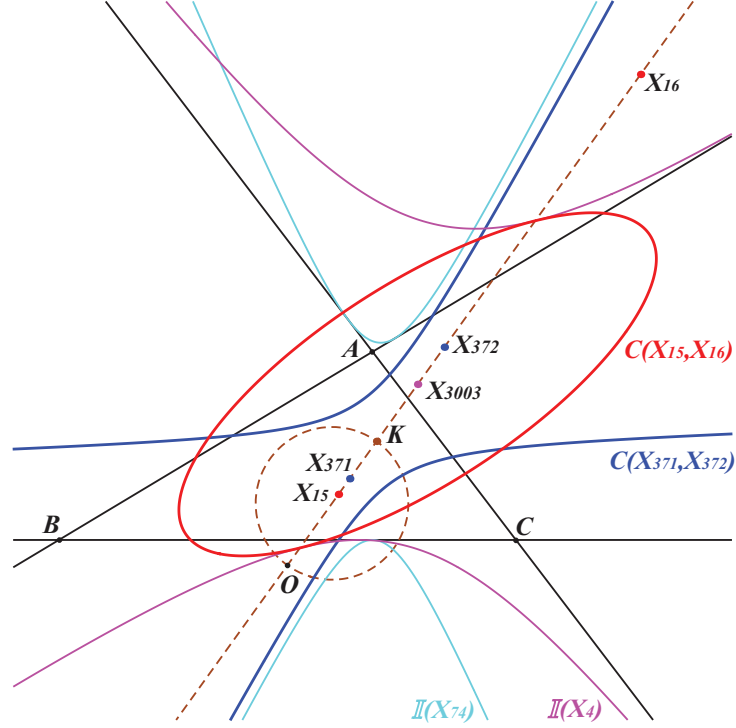


Figure 11: $\mathcal{C}(P, Q)$ and $\mathbb{I}(R)$ with P, Q on the Brocard axis

In case (2) one of these cubics K_a is the union of the conic C_a passing through B, C, P , the infinite points of the lines AB, AC and the line L_a passing through A and the foot of $\mathbb{P}(P)$ on BC . K_b and K_c are defined similarly.

When P does not lie on $\mathcal{L}^\infty$, these three cubics meet at seven points namely A, B, C, P and three other points Q_i giving three bicevian conics $\mathcal{C}(P, Q_i)$ which are the union of two parallel lines hence not having a proper center.

When P lies on $\mathcal{L}^\infty$, the three conics C_a, C_b, C_c decompose into $\mathcal{L}^\infty$ and a sideline of ABC . It follows that $\mathcal{C}(P, Q)$ and $\mathbb{I}(R)$ have always the same center when P and Q are two points on $\mathcal{L}^\infty$. This center is the barycentric product $P \times Q$ of P and Q . In other words, these conics are concentric and homothetic. Furthermore, $\mathcal{C}(P, Q)$ and $\mathbb{I}(R)$ have the same points at infinity which are those of the circum-conic whose perspector is $P^2 \times Q^2$. See figure 12 where $P = X_{30}$ and $Q = X_{512}$.

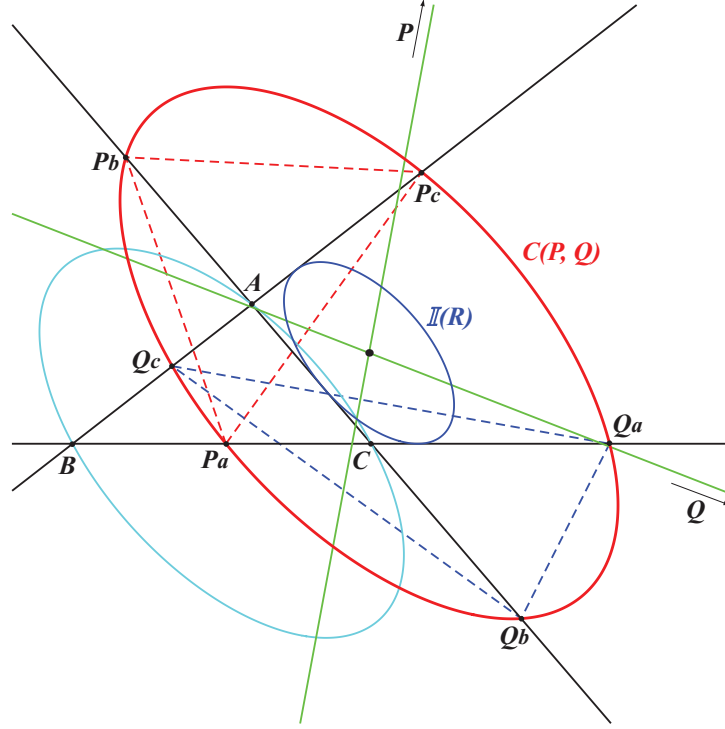


Figure 12: Concentric and homothetic conics $\mathcal{C}(P, Q)$ and $\mathbb{I}(R)$

3 Special bicevian conics

We have already seen that $\mathcal{C}(P, Q)$ is a circle if and only if P and Q are cyclocevian conjugates. We seek other special bicevian conics.

3.1 Bicevian parabola

3.1.1 Theorem

Theorem 8. $\mathcal{C}(P, Q)$ is a parabola if and only if Q lies on a quartic $\mathcal{Q}_P$ whose isotomic transform is a conic inscribed in the antimedial triangle. The complement of this conic is the in-conic whose perspector is the point $1/(u(v+w)) :::$, i.e. the isotomic conjugate of the complement of the isotomic conjugate of P .

3.1.2 Example

With $P = G$, Q_P is the isotomic conjugate of the Steiner circum-ellipse hence it is $\mathcal{L}^\infty$. Any parabola passing through the vertices of the medial triangle meets the sidelines of ABC again on three parallels passing through A, B, C . See figure 13.

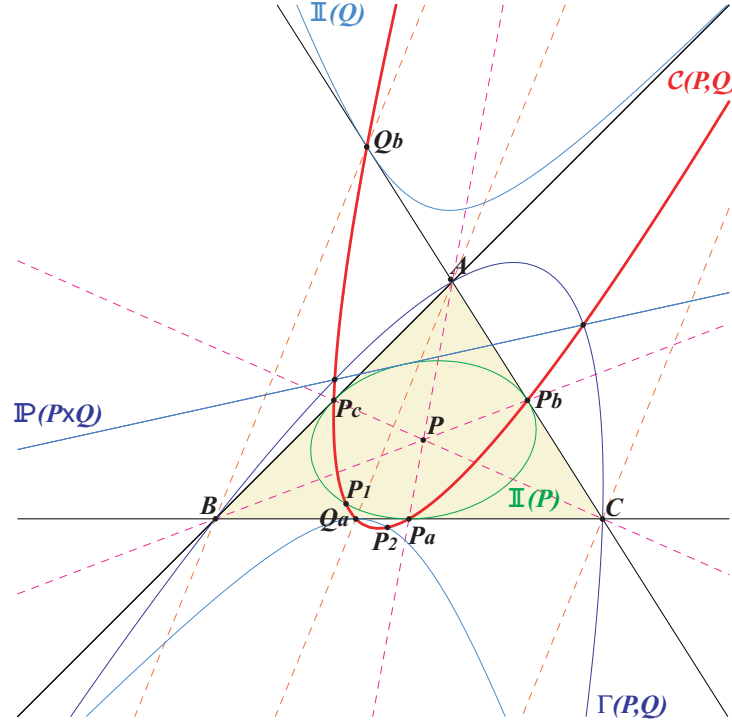


Figure 13: A parabola $\mathcal{C}(P, Q)$

3.2 Bicevian rectangular hyperbola

3.2.1 Theorem

Theorem 9. $\mathcal{C}(P, Q)$ is a rectangular hyperbola if and only if Q lies on the circum-conic Γ_P whose perspector is the point $S = S_B v(w+u) + S_C w(u+v) : \dots$

Naturally, this hyperbola contains the orthocenters of the triangles $P_aP_bP_c, Q_aQ_bQ_c$ and its center lies on the nine-point circles of these same triangles.

3.2.2 Construction of S

The anticomplement aS of S is the point $S_A u(v + w) ::$, barycentric product of the complement of the isotomic conjugate of P and the isotomic conjugate X_{69} of H .

3.2.3 Example

$\mathcal{C}(G, Q)$ is a rectangular hyperbola if and only if Q lies on the circumcircle and, in this case, $\mathcal{C}(G, Q)$ contains O . It is the complement of a rectangular circum-hyperbola. See figure 14.

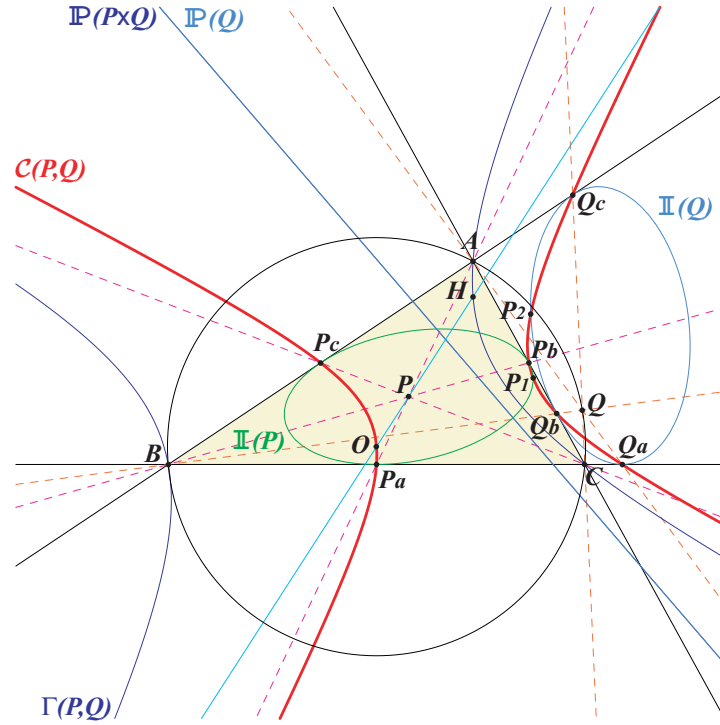


Figure 14: A rectangular hyperbola $\mathcal{C}(P, Q)$

4 The quadratic transformation $\varphi : P \mapsto S$

φ is a quadratic mapping with singular points A, B, C , with fixed points the four CPCC points. These four points lie on the Darboux and Lucas cubics, on the rectangular hyperbola through $X_5, X_6, X_{20}, X_{69}, X_{1498}$ whose asymptotes are parallel to those of the Jerabek hyperbola. They are the only points (apart H) such that their cevian and pedal circles coincide.

Examples of such points P, S :

X_i, X_j for $i, j = 1, 942 ; 2, 6 ; 3, 389 ; 4, 5 ; 7, 226 ; 8, 1210 ; 20, 4 ; 280, 1 ; 253, 2 ; 99, 525$.

$\varphi^{-1} : S \mapsto P$ is also a quadratic mapping with singular points the midpoints H_A, H_B, H_C of the altitudes. It is given by :

$$S = p : q : r \mapsto P = \frac{1}{-2\Delta^2(p+q+r) + qc^2S_C + rb^2S_B} ::$$

where Δ is the area of ABC .

Example : for any P on the circumcircle, S lies at infinity and Γ_P passes through G . In particular, with $P = X_{925}$, Γ_P is the Kiepert hyperbola hence, for any Q on the Kiepert hyperbola, $\mathcal{C}(X_{925}, Q)$ is a rectangular hyperbola passing through O .

φ maps any line to a conic through H_A, H_B, H_C . In particular, φ maps
– $\mathcal{L}^\infty$ to the one with center X_5 .
– the Euler line to the rectangular hyperbola through $X_4, X_5, X_6, X_{389}, X_{973}, X_{974}, X_{2574}, X_{2575}$.

There is one and only one line whose φ -transform is a circle, obviously the circumcircle of $H_AH_BH_C$. This line is perpendicular to the Euler line and has equation :

$$\sum_{\text{cyclic}} [a^6 + (b^2 - c^2)^2(2b^2 + 2c^2 - 3a^2)]x = 0.$$

φ maps any circum-conic to a line. For example, the circumcircle, the Kiepert hyperbola, the Jerabek hyperbola are mapped to $\mathcal{L}^\infty$, X_5X_6 , X_5X_{389} respectively.

5 Related cubics

Let P be a fixed point. The locus of X such that $P, X, \varphi(X)$ are collinear is a circum-cubic $\mathcal{K}(P)$ passing through the four CPCC points, P and $\varphi^{-1}(P)$.

All these cubics form a net of cubics generated by three nodal cubics $\mathcal{K}(A) = \mathcal{K}_a, \mathcal{K}_b, \mathcal{K}_c$. The equation of $\mathcal{K}_a$ is :

$$x(c^2y^2 - b^2z^2) + (S_By - S_Cz)yz = 0,$$

and those of $\mathcal{K}_b, \mathcal{K}_c$ similarly. Notice that the "sum" of these three nodal cubics is the Lucas cubic.

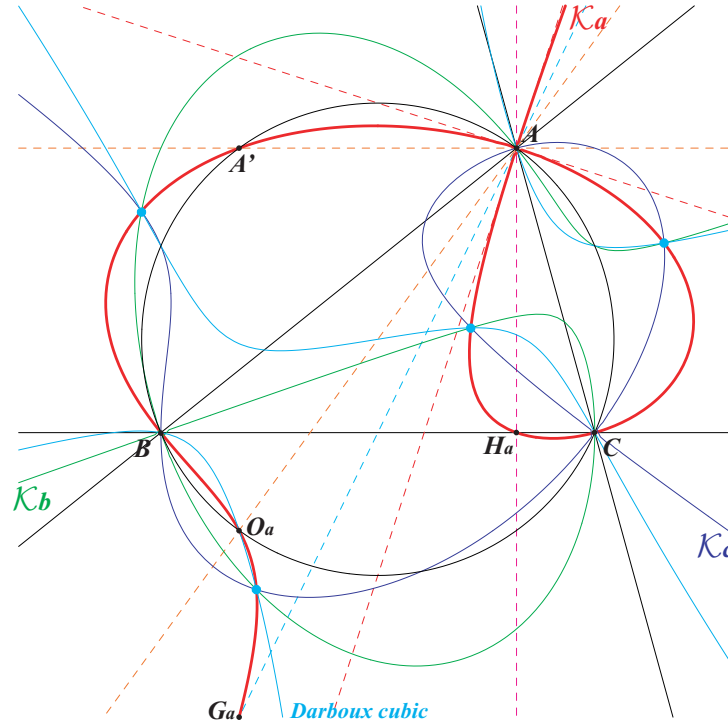


Figure 15: The three nodal cubics $\mathcal{K}_a, \mathcal{K}_b, \mathcal{K}_c$

More precisely, $\mathcal{K}_a$ is a nodal circum-cubic with node A and nodal tangents the bisectors at A . It contains G_a (anticomplement of A), the foot H_a of the altitude through A , the antipode O_a of A on the circumcircle, the

reflection A' of A in the perpendicular bisector of BC and the four CPCC points (light-blue points on the figure 15).

The cubic $\mathcal{K}(P)$ is a $p\mathcal{K}$ if and only if P lies on the Thomson cubic **K002**. Then, its pole is the cevian quotient G/P and also lies on the Thomson cubic. Its pivot is the anticomplement of the isogonal conjugate of P and lies on the Lucas cubic **K007**. When P traverses the Thomson cubic, the locus of $\varphi^{-1}(P)$ is the isotomic $p\mathcal{K}$ which is the isogonal transform of $p\mathcal{K}(X_{32}, X_{1498})$.

The following table gives a selection of such cubics denoted $p\mathcal{K}(i, j)$ for $p\mathcal{K}(X_i, X_j)$.

P	centers on the cubic	cubic
X_1	$X_1, X_8, X_{40}, X_{175}, X_{176}, X_{188}, X_{280}, X_{483}$	$p\mathcal{K}(9, 8) = \text{K199}$ Soddy-Nagel cubic
X_2	$X_2, X_4, X_7, X_8, X_{20}, X_{69}, X_{189}, X_{253}, X_{329}, X_{1032}, X_{1034}$	$p\mathcal{K}(2, 69) = \text{K007}$ Lucas cubic
X_3	$X_1, X_3, X_4, X_{20}, X_{40}, X_{64}, X_{84}, X_{1490}, X_{1498}, X_{2130}, X_{2131}$	$p\mathcal{K}(6, 20) = \text{K004}$ Darboux cubic
X_4	X_4, X_{20}	$p\mathcal{K}(1249, 4) = \text{K329}$
X_6	$X_2, X_3, X_6, X_{69}, X_{485}, X_{486}$	$p\mathcal{K}(3, 2) = \text{K168}$
X_9	$X_7, X_9, X_{84}, X_{329}, X_{366}, X_{1490}, X_{1659}$	$p\mathcal{K}(1, 329) = \text{K332}$
X_{57}	$X_1, X_7, X_{40}, X_{57}, X_{329}, X_{347}$	$p\mathcal{K}(223, 7) = \text{K333}$
X_{223}	$X_{174}, X_{189}, X_{223}$	$p\mathcal{K}(57, E_{623})$
X_{282}	$X_{84}, X_{189}, X_{282}, X_{1490}$	$p\mathcal{K}(E_{630}, 189)$
X_{1073}	$X_{64}, X_{253}, X_{1073}, X_{1498}$	$p\mathcal{K}(E_{382}, 253)$
X_{1249}	X_{253}, X_{1249}	$p\mathcal{K}(4, E_{624})$

Notes :

- E_{623} is the anticomplement of X_{282} .
- E_{630} is the complement of X_{1034} .
- E_{382} is the cevian quotient G/X_{1073} .
- E_{624} is the anticomplement of X_{1073} .

$\mathcal{K}(P)$ is a $n\mathcal{K}$ (in fact a $n\mathcal{K}_0$) if and only if P lies on a $n\mathcal{K}(X_6, X_{20})$ without any known center on it.

$\mathcal{K}(P)$ is circular if and only if P lies on $\mathcal{L}^\infty$. Then, its singular focus lies on the circle centered at X_{140} passing through X_{620} . For example, with $P = X_{30}$, the cubic is **K313**, the Darboux-Lucas circular cubic. It contains

$X_4, X_{13}, X_{14}, X_{20}, X_{30}, X_{1294}$. Its singular focus is the complement of X_{125} , the center of the Jerabek hyperbola. See figure 16.

$\mathcal{K}(P)$ is equilateral if and only if $P = X_{140}$ (nine-point center of the medial triangle) and it is **K268**.

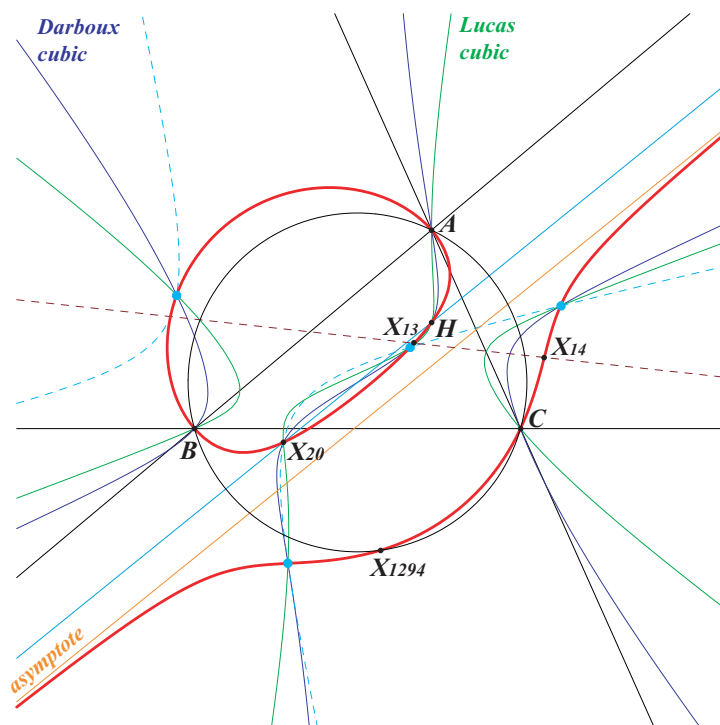


Figure 16: **K313**, the Darboux-Lucas circular cubic

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